Random Set Solutions to Stochastic Wave Equations

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Known Example – Tuned Mass Dampers

\[
M \begin{bmatrix} \ddot{x}_S \\ \ddot{x}_d \end{bmatrix} + C \begin{bmatrix} \dot{x}_S \\ \dot{x}_d \end{bmatrix} + K \begin{bmatrix} x_S \\ x_d \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{m_d}{m_s} \end{bmatrix} \ddot{x}_g
\]

Stochastic excitation \( \ddot{x}_g \)
Interval-valued coefficients in \( C, K \)
Response is a set-valued process

Interval valued trajectory and interval means w/o TMD:

![Graphs showing dim. less disp. and |Y| over time](image.png)
Known Example – Elastically Bedded Beam

Figure: a buried pipeline.


\[ EI w''''(x) + bc w(x) = q(x) \]

Load \( q(x) \) is a random field
Bedding parameter \( bc \) is an interval
Response is a set-valued process

Interval trajectory of bending moment, p-box for maximal bending moment:
New: SPDES, the Stochastic Wave Equation

The linear stochastic wave equation as a prototype of an SPDE:

\[
\begin{cases}
\partial_t^2 u_c - c^2 \Delta u_c = \dot{W}, & x \in \mathbb{R}^d, \ t \geq 0 \\
u_c|\{t < 0\} = 0
\end{cases}
\]

The Laplacian: \(\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_d}\).

Space-time white noise excitation \(\dot{W}\).

The solution process \(u_c = u_c(x, t, \omega)\).

Target: Uncertain propagation speed \(c\) as an interval \([c, \bar{c}]\).

Applications:

Acoustic waves in a medium under noisy disturbances.

Membrane under noisy excitation.

“A drum in the rain”.

Oberguggenberger/Wurzer

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Probability space \((\Omega, \Sigma, P)\). **White noise** is a generalized stochastic process with values in the space of distributions

\[
\Omega \to \mathcal{D}'(\mathbb{R}^{d+1}), \quad \omega \to \dot{W}(\omega)
\]

The solution \(\omega \to u_c(x, t, \omega)\) is a **stochastic process** with values in

- \(\mathcal{C}(\mathbb{R}^2), \ d = 1\) (classical)
- \(\mathcal{D}'(\mathbb{R}^{d+1}), \ d \geq 2\) (generalized)

**Resulting multifunction:**

\[
U(\omega) = \{u_c(\omega) : c \in [c, \bar{c}]\}
\]

with values in the power set of \(\mathcal{C}(\mathbb{R}^2)\), respectively \(\mathcal{D}'(\mathbb{R}^{d+1})\).

**Question:** Is \(U\) a **random set**? Implied by measurability of all

\[
U^{-}(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\}
\]

where \(B\) is any Borel subset of \(\mathcal{C}(\mathbb{R}^2)\), respectively \(\mathcal{D}'(\mathbb{R}^{d+1})\).
The classical case $d = 1$:

The map $c \rightarrow u_c(\omega)$ is continuous with values in $C(\mathbb{R}^2)$.

The image of $U(\omega)$ of $[c, \bar{c}]$ is compact.

Take a dense countable subset $c_1, c_2, \ldots$ of $[c, \bar{c}]$.

The sequence $u_{c_n}(\omega)$ is dense in $U(\omega)$ for every $\omega$.

Let $O$ be an open subset of $E$. Then

$$U^{-}(O) = \{\omega : U(\omega) \cap O \neq \emptyset\} = \bigcup_{n=1}^{\infty} \{\omega : u_{c_n}(\omega) \in O\}$$

is measurable.

$C(\mathbb{R}^2)$ is a **Polish space** (metrizable, complete, separable).

By the **Fundamental Measurability Theorem**, $U$ is a random set in $C(\mathbb{R}^2)$. 
**Higher Space Dimensions and New Results**

The generalized case \( d \geq 2 \):

Same argument, but \( \mathcal{D}'(\mathbb{R}^{d+1}) \) is **not a Polish space**.

**ANNOUNCEMENT 1:**

A **new measurability theorem for multifunctions with values in dual spaces** such as \( \mathcal{D}'(\mathbb{R}^{d+1}) \).

\( U \) is a random set also in space dimension \( d \geq 2 \).

**ANNOUNCEMENT 2:**

**Computation of upper and lower probabilities** of intervals \((a, b)\) of the set-valued solution \( U(x, t) \) at \((x, t)\) in \( d = 1 \), e.g.,

\[
\overline{P}(a, b)) = P(U(x, t) \cap (a, b) \neq \emptyset)
\]

This employs the observation that

\[
(r, \omega) \rightarrow \nu_r(\omega) = \frac{2}{t} u_{1/r}(x, t, \omega), \quad r > 0, \quad \nu_0(\omega) = 0
\]

is a Brownian motion.