Markov chains under nonlinear expectation

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(joint work with Robert Denk, Michael Kupper and Michael Röckner)

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Contents

1. Things that are not on the poster

2. Things that are on the poster
Contents

1 Things that are not on the poster

2 Things that are on the poster
Definition of nonlinear expectations

Let $(\Omega, \mathcal{F})$ be a measurable space. We denote by $L^\infty(\Omega, \mathcal{F})$ the space of all bounded measurable random variables $X: \Omega \to \mathbb{R}$. Throughout, let $M \subset L^\infty(\Omega, \mathcal{F})$ be a subspace that contains the constant gambles.

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A (nonlinear) pre-expectation \(\mathcal{E}\) is a functional \(\mathcal{E} : M \to \mathbb{R}\) with the following properties:

(i) Monotone: \(\mathcal{E}(X) \leq \mathcal{E}(Y)\) for all \(X, Y \in M\) with \(X \leq Y\).

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- *sublinear* (pre-)expectations, i.e.

\[
\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X) \quad \text{and} \quad \mathcal{E}(X + Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)
\]

for all \( X, Y \in M \) and \( \lambda > 0 \). In this case, \( \rho(X) := \mathcal{E}(-X) \) defines a coherent risk measure.
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- ...(pre-)expectations that are continuous from above or below, i.e.

  \[ E(X_n) \searrow E(X) \quad \text{or} \quad E(X_n) \nearrow E(X) \]

  for \((X_n)_{n \in \mathbb{N}} \subset M\) with \( X_n \searrow X \in M \) or \( X_n \nearrow X \in M \), respectively.
What do nonlinear expectations look like?

If $E$ is a linear expectation, then $E(X) = \int_X dP =: E_P(X)$, where $P$ is a finitely additive probability measure.

If $E$ is a sublinear expectation, then, $E(X) = \sup_{P \in \mathcal{P}} E_P(X)$, where $\mathcal{P}$ is nonempty a set of finitely additive probability measures.

If $E$ is a sublinear expectation, then, $E(X) = \sup_{P \in \mathcal{P}} E_P(X) - \alpha_P$, where $\mathcal{P}$ is a nonempty set of finitely additive probability measures and $\alpha_P \geq 0$ is a penalization for the model $P$.

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Markov chains under nonlinear expectation
06. 07. 2019 6 / 16
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Extension of pre-expectations (Denk-Kupper-N. (2018))

We consider two extension procedures for pre-expectations to an expectation:

1) Extension without continuity assumptions: For $X \in L^\infty(\Omega, \mathcal{F})$, let $\hat{E}(X) := \inf \{ E(Y) \mid Y \in M, Y \geq X \}$.

Inspired by Kantorovich's extension of positive linear functionals, closely linked to the idea of superhedging (≈ NFL), preserves convexity and sublinearity, the maximal extension and representation in terms of finitely additive measures.

2) Extension if $E$ is continuous from above: For $X \in L^\infty(\Omega, \mathcal{F})$, let $E(X) := \sup \{ \inf \{ E(X_n) \mid (X_n) \subseteq M, X_n \geq X_{n+1} \} \}$.

Inspired by Choquet's theorem on capacitability and outer measures, again, linked to superhedging (≈ NFLVR, Delbaen-Schachermayer (1994)), preserves convexity and sublinearity, uniqueness (in a certain sense) and representation in terms of countably additive measures.
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2) Extension if $E$ is continuous from above: For $X \in L^\infty(\Omega, \mathcal{F})$, let

$$\overline{E}(X) := \sup \left\{ \inf_{n \in \mathbb{N}} E(X_n) \mid (X_n)_{n \in \mathbb{N}} \subset M, X_n \geq X_{n+1}, \inf_{n \in \mathbb{N}} X_n \leq X \right\}. $$

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Stochastic Processes under nonlinear expectations

The extension procedures from the last slide can be used to derive an imprecise version Kolmogorov's theorem on the existence of stochastic processes. This reduces the existence of Markov process to certain properties of a family of transition operators (so-called regular kernels) \((\mathcal{E}_{s,t})_{0 \leq s < t}\). The construction of the transition operators is inspired by Nisio (1976). Main examples are:

- A Brownian Motion with imprecise drift \(\mu \in [\mu, \mu]\) (≈ BSDEs, El Karoui-Peng-Quenez (1997), Coquet et al. (2002))
- A Brownian Motion with imprecise volatility \(\sigma \in [\sigma, \sigma]\) (≈ 2BSDEs, Peng (2007, 2008), Denis-Hu-Peng (2011), Soner-Touzi-Zhang (2011a, 2011b))
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Contents

1 Things that are not on the poster

2 Things that are on the poster
Q-matrices

Definition

A matrix \( q = (q_{ij})_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d} \) is called a \( Q \)-matrix if it satisfies the following:

(i) \( q_{ii} \leq 0 \) for all \( i \in \{1, \ldots, d\} \),
(ii) \( q_{ij} \geq 0 \) for all \( i, j \in \{1, \ldots, d\} \) with \( i \neq j \),
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We say that a (possibly nonlinear) map \( \mathcal{Q}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \) satisfies the positive maximum principle (PMP) if for \( v = (v_1, \ldots, v_d)^T \in \mathbb{R}^{d} \) and \( i \in \{1, \ldots, d\} \) the following implication holds:

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  v_i = \max_{j=1,\ldots,d} v_j \quad \Longrightarrow \quad (\mathcal{Q}v)_i \leq 0
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\[\rightarrow q \in \mathbb{R}^{d \times d} \text{ is a Q-matrix if and only if it satisfies the PMP and } 1 \in \ker q.\]
Examples for Q-matrices

Examples for Q-matrices are:

\[ d = 3 : \quad q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \iff \quad \mu \partial_x \]

\[ d = 3 : \quad q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad \iff \quad \sigma^2 \partial_{xx} \]

\[ d = 4 : \quad q = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \quad \iff \quad \sigma^2 \partial_{xx} \]
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Overview

\{Markov chains\} \leftrightarrow \{Markov semigroups\}

\{Q-matrices\} \leftrightarrow \{q \text{ linear PMP, } q^1=0\}
Q-operators: A generalization of Q-matrices

We now want to generalize the concept of a Q-matrix to a nonlinear setup.

Definition (Just for the sublinear case)

A (possibly nonlinear) map $\mathcal{Q}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called a Q-operator if the following conditions are satisfied:

(i) $(\mathcal{Q}(e_i))_i \leq 0$ for all $i \in \{1, \ldots, d\}$,

(ii) $(\mathcal{Q}(-e_i))_j \leq 0$ for all $i, j \in \{1, \ldots, d\}$ with $i \neq j$,

(iii) $\mathcal{Q}(m1) = 0$ for all $m \in \mathbb{R}$, where $1 := (1, \ldots, 1)^T \in \mathbb{R}^d$. 

One immediately sees that a linear Q-operator is a Q-matrix and vice versa.

Example: For $d = 3$ and $0 < \sigma \leq \sigma$ we consider the mapping $\mathcal{Q}v := \sup_{\sigma \in [\sigma, \sigma]} \sigma^2 \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} v \approx \sup_{\sigma \in [\sigma, \sigma]} \sigma^2 \partial_{xx} v$.
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A (possibly nonlinear) map $Q: \mathbb{R}^d \to \mathbb{R}^d$ is called a $Q$-operator if the following conditions are satisfied:

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→ One immediately sees that a linear $Q$-operator is a $Q$-matrix and vice versa.

Example: For $d = 3$ and $0 < \underline{\sigma} \leq \overline{\sigma}$ we consider the mapping

$$Qv := \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \left( \begin{array}{ccc} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{array} \right) v \approx \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \partial_{xx} v$$
Theorem (Main result, the sublinear case)

Let $\mathcal{Q}: \mathbb{R}^d \to \mathbb{R}^d$ be a mapping. Then the following statements are equivalent:

(i) $\mathcal{Q}$ is a sublinear $Q$-operator.

(ii) $\mathcal{Q}$ is sublinear, satisfies the PMP and $\mathcal{Q}(m1) = 0$ for all $m \in \mathbb{R}$.

(iii) There exists a set $\mathcal{P} \subset \mathbb{R}^{d \times d}$ of $Q$-matrices such that

$$\mathcal{Q}u_0 = \sup_{q \in \mathcal{P}} q u_0 \quad \text{for all } u_0 \in \mathbb{R}^d.$$ 

(iv) There exists a sublinear Markov semigroup $(S(t))_{t \geq 0}$ such that $u(t) := S(t)u_0$ defines the unique solution $u \in C^1([0, \infty); \mathbb{R}^d)$ to the initial value problem

$$u'(t) = \mathcal{Q}u(t) \quad \text{for all } t \geq 0, \quad u(0) = u_0.$$ 

(ODE)

(v) There exists a sublinear Markov chain $(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t \geq 0})$ such that

$$\mathcal{Q}u_0 = \lim_{h \searrow 0} \frac{\mathcal{E}(u_0(X_h)) - u_0}{h} \quad \text{for all } u_0 \in \mathbb{R}^d.$$
Some remarks and conclusions

The previous theorem gives an axiomatization of the generators of sublinear Markov chains. The transition operators are given by $S(t)u_0 = E(u_0(X_t))$ and have an "explicit" primal and dual representation.

Solutions to (ODE) remain bounded. Therefore, a Picard iteration can be used for numerical computations and the convergence rate (depending on the size of the initial value $u_0$) can be explicitly computed. Other numerical methods such as Runge-Kutta methods can also be applied.

By solving (ODE) (for example with Euler's method), we can compute "prices" for European contingent claims of the form $u(t) = E(u_0(X_t))$ under model uncertainty. More precisely, $E(u_0(X_t)) \approx (I + tnQ)^nu_0$. 

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- The previous theorem gives an axiomatization of the generators of sublinear Markov chains.
- The transition operators are given by $S(t)u_0 = \mathcal{E}(u_0(X_t))$ and have an “explicit” primal and dual representation.
- Solutions to (ODE) remain bounded. Therefore, a Picard iteration can be used for numerical computations and the convergence rate (depending on the size of the initial value $u_0$) can be explicitly computed. Other numerical methods such as Runge-Kutta methods can also be applied.
- By solving (ODE) (for example with Euler’s method), we can compute “prices” for European contingent claims of the form

\[ u(t) = \mathcal{E}(u_0(X_t)) \]

under model uncertainty. More precisely,

\[ \mathcal{E}(u_0(X_t)) \approx \left( I + \frac{t}{n} \mathcal{Q} \right)^n u_0. \]
Thank you very much for your attention and see you at the poster! :-)

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Markov chains under nonlinear expectation
06. 07. 2019 16 / 16