Hitting Times and Probabilities for Imprecise Markov Chains

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Markov Chains

Stochastically evolving dynamical system with uncertain state $X_n$

- Time $n \in \mathbb{N}_0$ (*discrete time* model)
- Finite state space $\mathcal{X}$

A stochastic process $P$ is called a *Markov chain* if

$$P(X_{n+1} = x_{n+1} \mid X_{0:n} = x_{0:n}) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

A Markov chain is called *homogeneous* if, moreover,

$$P(X_{n+1} = y \mid X_n = x) = P(X_1 = y \mid X_0 = x)$$
A \textit{transition matrix} $T$ is an $|\mathcal{X}| \times |\mathcal{X}|$ matrix that is row-stochastic:

\[
\sum_{y \in \mathcal{X}} T(x, y) = 1 \text{ and } T(x, y) \geq 0
\]

Such a $T$ determines a homogeneous Markov chain $P$ for which

\[
P(X_{n+1} = y \mid X_n = x) = T(x, y) \quad \text{for all } x, y \in \mathcal{X} \text{ and } n \in \mathbb{N}_0.
\]
What if we don’t know $T$?

Or: what if Markov assumption is unwarranted?

⇒ Instead use an *imprecise* Markov chain
Imprecise Markov Chains

Parameterised by a set $\mathcal{T}$ of transition matrices.
- $\mathcal{T}$ must satisfy some technical closure properties.

Inferences are the lower and upper expectations of quantities of interest.

These depend on the type of imprecise Markov chain!
Imprecise Markov Chains

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For the set $\mathcal{P}^H_\mathcal{T}$ of homogeneous Markov chains with transition matrix $T$ in $\mathcal{T}$,

$$E^H_\mathcal{T}[\cdot | \cdot] = \inf_{P \in \mathcal{P}^H_\mathcal{T}} E_P[\cdot | \cdot] \quad \text{and} \quad \bar{E}^H_\mathcal{T}[\cdot | \cdot] = \sup_{P \in \mathcal{P}^H_\mathcal{T}} E_P[\cdot | \cdot]$$

What other types are there?
Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathcal{T}$. 

- Set of Markov chains such that for all $n \in \mathbb{N}_0$ there is some $T \in \mathcal{T}$ for which

$$P(X_{n+1} = x_{n+1} | X_n = x_n) = T(x_n, x_{n+1}).$$

Called a Markov set chain, or an imprecise Markov chain under strong independence.

- Set of general stochastic processes "compatible" with $T$.

Always some $T \in \mathcal{T}$ such that

$$P(X_{n+1} = x_{n+1} | X_0: n = x_0: n) = T(x_n, x_{n+1}),$$

but can be different $T$ for each $x_0: n$.

Called an imprecise Markov chain under epistemic irrelevance.

Game-theoretic probability model with local uncertainty models described by $T$. 

$\mathbb{E}_H^\mathcal{F} [\cdot | \cdot]$
Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathcal{T}$.

- *Game-theoretic probability model* with local uncertainty models described by $\mathcal{T}$.

\[
\mathbb{E}^V_{\mathcal{T}}[\cdot | \cdot] \leq \mathbb{E}^H_{\mathcal{T}}[\cdot | \cdot]
\]
Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix \( T \in \mathcal{T} \).

- Set of general stochastic processes “compatible” with \( \mathcal{T} \).
  Always some \( T \in \mathcal{T} \) such that
  \[
P(X_{n+1} = x_{n+1} \mid X_{0:n} = x_{0:n}) = T(x_n, x_{n+1}),
\]
  but can be different \( T \) for each \( x_{0:n} \).
  Called an imprecise Markov chain under epistemic irrelevance.

- Game-theoretic probability model with local uncertainty models described by \( \mathcal{T} \).

\[
E_{\mathcal{T}}^V[\cdot \mid \cdot] \leq E_{\mathcal{T}}^I[\cdot \mid \cdot] \leq E_{\mathcal{T}}^H[\cdot \mid \cdot]
\]
Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathcal{T}$.
- Set of Markov chains such that for all $n \in \mathbb{N}_0$ there is some $T \in \mathcal{T}$ for which
  \[ P(X_{n+1} = x_{n+1} | X_n = x_n) = T(x_n, x_{n+1}). \]
  Called a Markov set chain, or an imprecise Markov chain under strong independence.
- Set of general stochastic processes "compatible" with $\mathcal{T}$.
  Always some $T \in \mathcal{T}$ such that
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  but can be different $T$ for each $x_{0:n}$.
  Called an imprecise Markov chain under epistemic irrelevance.
- Game-theoretic probability model with local uncertainty models described by $\mathcal{T}$.

$$
\mathbb{E}^V_{\mathcal{T}}[\cdot | \cdot] \leq \mathbb{E}^I_{\mathcal{T}}[\cdot | \cdot] \leq \mathbb{E}^M_{\mathcal{T}}[\cdot | \cdot] \leq \mathbb{E}^H_{\mathcal{T}}[\cdot | \cdot]
$$
Lower and Upper Expected Hitting Times

Given a fixed set $A \subseteq \mathcal{X}$ of states:

*How long will it take before the system visits an element of $A$?*

What is $\mathbb{E}_P[H_A | X_0]$, where $H_A$ is the number of steps before $A$ is visited?

What can we say about this for the various types of imprecise Markov chains?
Lower and Upper Expected Hitting Times

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**Theorem**

$$
\mathbb{E}^V[ H_A | X_0 ] = \mathbb{E}^I[ H_A | X_0 ] = \mathbb{E}^M[ H_A | X_0 ] = \mathbb{E}^H[ H_A | X_0 ]
$$

(and similarly for the upper expected hitting time)