First steps towards an imprecise Poisson process

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Poisson-events

We are interested in the repeated occurrences of a Poisson-event over time, but the exact time instants of these occurrences are uncertain to us; for example, the arrival of a customer to some queue. For every time instant $t$, we let $X_t$ be the number of Poisson-events that have occurred up to $t$; hence, $X_t$ is non-decreasing with $t$.

The Poisson process in particular

For a Poisson process, one additionally assumes that the transition probabilities $P$ are Markov:

\[
P(X_{t+1} = y \mid X_t = x, X_0 = x_0) = P(X_{t+1} = y \mid X_t = x);
\]

PP1. are Markov:

PP2. only depend on the length of the time interval:

\[
P(X_{t+1} = y \mid X_t = x) = P(X_{t} = y \mid X_0 = x);
\]

PP3. only depend on the number of occurred events in the time interval:

\[
P(X_{t} = y \mid X_0 = x) = P(X_{t} = y - x \mid X_0 = 0) = 0.
\]

It is well-known that a Poisson process is uniquely characterised by a single parameter: the rate $\lambda$.

Set of consistent counting processes

Another option is to consider the set $\mathcal{P}_{\text{cp}}$ of all counting processes $P$ that are consistent with the rate interval $[\underline{\lambda}, \overline{\lambda}]$, in the sense that

\[
\underline{\lambda} + o(\Delta) \leq P(X_{t+1} = x+1 \mid X_t = x, X_0 = x_0) \leq \overline{\lambda} + o(\Delta).
\]

As every Poisson process is a counting process, this set is more general than the set of Poisson processes:

\[
\mathcal{P}_{\text{pp}} \subseteq \mathcal{P}_{\text{cp}};
\]

this inclusion is in fact strict! We let $\mathcal{E}_{\text{cp}}(\cdot \mid \cdot)$ denote the lower envelope of the expectations $\mathcal{E}_{\text{pp}}(\cdot \mid \cdot)$ with respect to all $P$ in $\mathcal{P}_{\text{cp}}$. Then clearly,

\[
\mathcal{E}_{\text{cp}}(\cdot \mid \cdot) \leq \mathcal{E}_{\text{pp}}(\cdot \mid \cdot) \leq \mathcal{E}_{\text{cp}}(\cdot \mid \cdot).
\]

At first sight, computing the lower expectation $\mathcal{E}_{\text{cp}}$ requires the explicit construction of and subsequent optimisation over the set $\mathcal{P}_{\text{cp}}$; a non-trivial optimisation problem! However, we show that

\[
\mathcal{E}_{\text{cp}}(f(X_{t+1}) \mid X_t = x, X_0 = x_0) = \mathcal{E}_{\text{cp}}(f(X_t) \mid x),
\]

a tractable optimisation problem!

From this, it follows that—quite remarkably—the lower expectation $\mathcal{E}_{\text{cp}}(\cdot \mid \cdot)$ satisfies the imprecise versions of (PP1)–(PP3) as well as Equations (1) and (2), just like $\mathcal{E}_{\text{pp}}(\cdot \mid \cdot)$.

In general, we model our beliefs by specifying the transition probabilities

\[
P(X_{t+1} = y \mid X_t = x, X_0 = x_0, \ldots, X_{t_i} = x_i),
\]

where $t_i, \ldots, t_n$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ and $x_1, \ldots, x_{t_i} = x$ is a non-decreasing sequence in $\mathbb{R}_{\geq 0}$.

What if we only know that the rate $\lambda$ belongs to the rate interval $[\underline{\lambda}, \overline{\lambda}]$?

Let $\mathcal{L}$ be the real vector space of all bounded real-valued functions on $\mathbb{R}_{\geq 0}$. Essential to our approach is the generator $Q : \mathcal{L} \to \mathcal{L}$, defined as

\[
Q(f)(x) := \min_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \lambda f(x + 1) - \lambda f(x).
\]

We show that

\[
\Phi_{\lambda_n} := \left( 1 + \frac{\Delta}{n} \right)^n
\]

converges to a transformation on $\mathcal{L}$ in the limit for $n \to +\infty$. Hence, we can define

\[
\mathcal{L}_\Delta := \lim_{n \to +\infty} \Phi_{\lambda_n}.
\]

For functions $f$ such that

\[
f(y) = f(y) e^{\frac{\lambda y}{n}} + f(y) e^{\frac{\lambda y}{n}}
\]

can determine $\mathcal{L}_\Delta f(x)$ by means of transformations on the vector space of real-valued functions on the finite set $\{y \in \mathbb{R}_{\geq 0} : y \leq \Delta\}$.

This is extremely useful in practice because, for general bounded functions $f$,

\[
\lim_{n \to +\infty} \mathcal{L}_n f(x) = \mathcal{L}_\Delta f(x).
\]

Similar limit techniques also work for functions that are only bounded below.

See arXiv: 1905.05734 for all details!

Counting processes in general

For a counting process, we assume that

\[
P(X_0 = 0) = 1;
\]

CP1. two Poisson-events can not occur at the same time:

\[
P(X_{t+1} \geq x + 2 \mid X_t = x, X_0 = x) = o(\Delta).
\]

Set of Poisson processes

One option is to consider the set $\mathcal{P}_{\text{pp}}$ of all Poisson processes with a rate that belongs to the rate interval $[\underline{\lambda}, \overline{\lambda}]$.

We let $\mathcal{E}_{\text{pp}}(\cdot \mid \cdot)$ denote the lower envelope of the expectations $\mathcal{E}_{\text{pp}}(\cdot \mid \cdot)$ with respect to all $P$ in $\mathcal{P}_{\text{pp}}$. Clearly, we can compute this lower expectation by means of a one-parameter optimisation problem.

This lower expectation $\mathcal{E}_{\text{pp}}(\cdot \mid \cdot)$ satisfies imprecise versions of (PP1)–(PP3):

1. Markovianity:

\[
\mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_t = x, X_0 = x_0) = \mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_t = x);
\]

2. time-homogeneity:

\[
\mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_t = x, X_0 = x_0) = \mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_0 = x_0);
\]

3. state-homogeneity:

\[
\mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_0 = x_0) = \mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_0 = x_0) + \Delta.
\]

Furthermore,

\[
\mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_0 = x_0) = \mathcal{E}_{\text{pp}}(f(X_{t+1}) \mid X_0 = x_0) + \Delta.
\]

However, assuming (PP1)–(PP3) is not always justified!

Numerical example

Below, we have depicted tight lower and upper bounds—with respect to both sets—on the probability of having no Poisson-event or a single Poisson-event in a time period of length $\Delta$ for the rate interval $[\underline{\lambda}, \overline{\lambda}] = [1, 2]$.