We will show that even convexity follows also from the two axioms Archimedean, then for any gamble \( D \)

\[ D \ni 0 \notin D; \]

\[ D_f \ni \text{if } f < g \text{ then } f \in D; \]

\[ D_f \ni \text{if } f < D \text{ then } f \not\in D; \]

\[ D_f \ni \text{if } f + g \not\in D. \]

A coherent set of desirable gambles is a convex cone that includes the gambles \( f > 0 \) and has nothing in common with \( \{0\} \) and the gambles \( f < 0 \). Note that Axiom \( D_f \) only requires that point-wise positive gambles be desirable: we do not require admissibility or weak dominance, even in finite state spaces.

**Probabilities**

Given a probability mass function \( p \) with corresponding expectation operator \( E_p \), the set

\[ D_p := \{ f \in \mathcal{L} : E_p(f) > 0 \}, \]

is coherent. It is the smallest coherent set of desirable gambles whose lower prevision is equal to \( E_p \). We have that every set of desirable gambles \( D \) is in fact a \( D_p \) if and only if it is an open semispace that includes the gambles \( f > 0 \). We say that a set of desirable gambles \( D \) is *represented* by a set \( K \) of probability mass functions when \( D = \bigcap_{p \in K} D_p \).

When is a coherent set of desirable gambles represented by a set of probability mass functions?

If \( D \) is represented by a set of probabilities, then the largest \( K \) that represents \( D \) is convex, but not necessarily closed. [Fabio Cozman, *Evenly convex credal sets*, IJAR 2018] pointed out that \( D \) is represented by \( K \) if and only if \( D \) is *evenly convex*—meaning that it is an arbitrary intersection of affine open semi-spaces—and gives an elegant equivalent requirement in terms of gambles.

**Examples**

Taken from [Fabio Cozman, *Evenly convex credal sets*, IJAR 2018]:

- evenly convex
- not evenly convex
- not evenly convex
- evenly convex
- evenly convex
- not evenly convex
- not evenly convex
- not evenly convex
- evenly convex
- evenly convex
- not evenly convex

We will show that even convexity follows also from the two axioms SSK–Archimedeanity and SSK–Extension from [Seidenfeld, Schervish, and Kadane, *A representation of partially ordered preferences. The Annals of Statistics 1995*].

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**SSK–Archimedeanity**

The requirement SSK–Archimedeanity from [Seidenfeld, Schervish, and Kadane, A representation of partially ordered preferences. The Annals of Statistics 1995], expressed for gambles, is (with the aid of a lemma) equivalent to:

**Definition (SSK–Archimedeanity)** A set of desirable gambles is called SSK–Archimedean when \( D + a(D) \leq D \), for any set of gambles \( A \) and \( B \), their addition is defined as \( A + B := \{ a + b : a \in A, b \in B \} \).

**SSK–Archimedeanity takes core of the internal part of the boundary.**

**Proposition** Consider any coherent set of desirable gambles \( D \). If \( D \) is SSK–Archimedean, then for any gamble \( f \) on a linear part of the boundary of \( D \), \( f \not\in D \) then the whole interior of this linear part of the boundary is contained in \( D \).

**SSK–Extension**

For the requirement SSK–Extension from [Seidenfeld, Schervish, and Kadane, A representation of partially ordered preferences. The Annals of Statistics 1995], we need two new notions:

**Precluded desirability** A gamble \( f \) is called precluded from being desirable when

\[ \neg f \in c(D). \]

**Precluded \( p \)-indifference** A gamble \( f \) is called precluded from being \( p \)-indifferent to \( 0 \) if assuming that \( f \) is a probability mass function such that \( \int f(x) = \lambda \) will be incompatible with \( D \), in the sense that \( E_p(f) \leq 0 \) for some \( p \in D \).

**Definition (SSK–Extension)** For any coherent set of desirable gambles \( D \), its SSK–Extension \( D^* \) is given by

\[ D^* := \{ f \in \mathcal{L} : f \not\in D \text{ or } (\neg f \text{ is precluded from being desirable and } f \text{ is precluded from being } p\text{-indifferent to } 0) \}. \]

**SSK–Archimedeanity and SSK–Extension take care of the boundary.**

**Proposition** Consider any coherent set of desirable gambles \( D \) that is SSK–Archimedean. Then \( D^* \) is coherent and evenly convex.

**Ice Cream Cone Corollary** Consider any coherent set of desirable gambles \( D \) that is SSK–Archimedean. Then, \( D \) contains all its extreme non-exposed points if and only if \( D \) is evenly convex. This extends Theorem 16 of [Fabio Cozman, *Evenly convex credal sets*, IJAR 2018].