The Ergodic conundrum

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1. Motto

“No probability without ergodicity”

Nassim Taleb, Skin in the game (2018)

2. Main questions

○ if \( f : M \to M \) is a map defined on a compact topological (metric) space \( M \) and \( A \subset M \). Can we give mathematical sense to the questions:

\[ \text{Given a point } x \in M \text{ what is the probability of that point visiting the set } A? \]

Without knowing the initial state what is the probability of the set \( A \) being visited?

It is not the aim of this work to give a definitive answer to these questions. Rather we will try to make them more IMPRECISE...

3. An application of Birkhoff’s theorem

○ probability measure \( \mu \) defined on a \( \sigma \)-algebra \( B \subset \mathcal{P}(M) \) is invariant under transformation \( f \) if \( \mu(A) = \mu(f^{-1}(A)) \) for every \( A \in \mathcal{B} \).

\[
\frac{1}{n} \sum_{k=0}^{n-1} f_k (x) \xrightarrow{n \to \infty} \text{freq}_A(x)
\]

for \( n \)-almost every point \( x \) and moreover

\[
\mu(A) = \int \text{freq}_A(x) \, d\mu
\]

\( \nabla \) the set of points which do not have converging frequencies of visits to a given set is neglected from the point of view of the given measure

\( \nabla \) the probability of visiting a set cannot be interpreted as an average of the convergent frequencies because this average is calculated with respect to \( \mu \)

SO, the Ergodic theorem of Birkhoff should not be used as a means of interpreting the probability of a physical system visiting a set.

4. Not all sets are welcome!

○ point \( x \) which is not eventually periodic

○ sequence \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in \{0,1\}^* \)

\[ A_\sigma = \{ f^k(x) : \sigma_k = 1 ; k \in \mathbb{N} \} \]

\[ \gamma : \{ k : f^k(x) \in A_\sigma \} = \{ k : \sigma_k = 1 \} \]

.: Any sequence of visits can be produced for any such \( x \).

SO, consider open and connected sets? e.g. open balls in the case of a metric space

5. Upper and Lower operators

\( \diamond x \in \mathcal{M} \) and \( \phi : \mathcal{M} \to \mathbb{R} \) bounded function

\[ L_x(\phi) := \lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi( f^k(x) ) \]

\[ U_x(\phi) := \lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi( f^k(x) ) \]

\( L_x \) and \( U_x \) are both:

- monotone: \( \phi_1 \leq \phi_2 \) implies \( \Gamma(\phi_1) \leq \Gamma(\phi_2) \)

- positive homogeneous: \( \Gamma(\lambda \phi) = \lambda \Gamma(\phi) \), \( \lambda > 0 \)

- translation invariant: \( \Gamma(\phi \circ c) = \Gamma(\phi) + c, c \in \mathbb{R} \)

- \( f \)-invariant: \( \Gamma(\phi \circ f) = \Gamma(\phi) \)

- \( L_x \) is superadditive:

\[ L_x(\phi_1 + \phi_2) \geq L_x(\phi_1) + L_x(\phi_2) \]

- \( U_x \) is subadditive:

\[ U_x(\phi_1 + \phi_2) \leq U_x(\phi_1) + U_x(\phi_2) \]

\[ \ell_x(\phi) := \inf_{n \in \mathbb{N}} \frac{1}{n} \sup_{k=1}^{n} \phi( f^k(x) ) \]

\[ u_x(\phi) := \sup_{n \in \mathbb{N}} \frac{1}{n} \inf_{k=1}^{n} \phi( f^k(x) ) \]

\[ \ell_x \] and \( u_x \) are set functions satisfying:

- \( \ell_x(0) = u_x(0) = 0 \) and \( \ell_x(M) = u_x(M) = 1 \)

- \( u_x \) is the dual set function of \( \ell_x \) that is \( \ell_x(A) + u_x(A^c) = 1 \), (where \( A^c \) is the complementary set of \( A \))

- if \( A \subset B \) then \( \ell_x(A) \leq \ell_x(B) \) and \( u_x(A) \leq u_x(B) \)

- \( f \)-invariance:

\[ \ell_x(\phi) = \ell_y(\phi \circ f) \] and \( u_x(\phi) = u_y(\phi \circ f) \)

6. Modularity of \( \ell_x \) and \( u_x \)

By a well known representation result (see [1]) there exist two sets of finitely-additive measures \( F^+ \) and \( F^- \) such that

\[
\ell_x(\phi) = \inf_{\mu \in F^+} \left\{ \int \phi \, d\mu \right\}
\]

\[
\ell_x(\phi) = \sup_{\nu \in F^-} \left\{ \int \phi \, d\nu \right\}
\]

Using a characterisation result of sub-(resp. super-)modular set functions (see [2]) we conclude that

\[ \ell_x \] is supermodular:

\[ \ell_x(A \cup B) + \ell_x(A \cap B) \geq \ell_x(A) + \ell_x(B) \]

\( u_x \) is submodular:

\[ u_x(A \cup B) + u_x(A \cap B) \leq u_x(A) + u_x(B) \]

Note: For any set function \( \nu \) represented as a lower envelope of a family of capacities, \( F_x \), we have that \( F_x \) can be chosen as \( \text{core}(\nu) \) where

\[ \text{core}(\nu) = \{ P : P \geq \nu \} \]

7. Probability... imprecisely

○ if the phase space contains two (open) forward-invariant sets \( A \) and \( B \) such that \( A \cap B = \emptyset \) then for both sets we have that \( \text{Pr}(A) = \text{Pr}(B) = 0 \), \( [0,1] \) which is empty information. SO, there must be some form of indecomposability or restriction of phase space associated with this definition of probability.

8. Special case

○ \( \lambda \) is a \textit{a priori} defined probability on a \( \sigma \)-algebra of the space \( M \) (e.g., a normalised Liouville measure)

Alternatively, we can define

\[
\text{Pr}(A) := [\int_M \ell_x(A) \, d\lambda, \int_M u_x(A) \, d\lambda]
\]

\( \Delta \) if \( \lambda \) is \( f \)-invariant itself then

\[
\int_M \ell_x(A) \, d\lambda = \int_M u_x(A) \, d\lambda = \lambda(A)
\]

We can also define the “distribution functions”

\[ G_{\ell_x}(s) = \lambda(x \in A : \ell_x(A) > s) \]

\[ G_{u_x}(s) = \lambda(x \in A : u_x(A) > s) \]

This allows for a representation of the “probability” of visiting a set \( A \) as a set of functions, i.e.

\[ \{ g : [0,1] \to [0,1] : G_{\ell_x}(s) \leq g(s) \leq G_{u_x}(s) \} \]

9. In a very special case...

...we can prove a version of the Khintchine recurrence theorem:

Let \( \nu \) be an \( f \)-invariant set function such that \( \nu(A) = \int \phi \, d\nu \) and \( \nu(v) \) contained in the set of invariant probabilities of \( f \). Then for every \( 0 \leq \alpha < 1 \) the set

\[ \{ x : \nu(A \cap f^{-n}(A)) > \alpha \nu(A) \} \]

is syndetic.

10. Poincaré recurrence generalised

Given a \( f \)-invariant superadditive capacity \( \nu \) (e.g., \( \ell_x \)) defined on a system \( S \subset \mathbb{R}^d \) on which \( f \) is measurable then for every measurable \( A \) with \( \nu(A) > 0 \) the set of points that return infinitely often to \( A \) has measure \( \nu(A) \).

References


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